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Pressure gradient influence on global lymph flow

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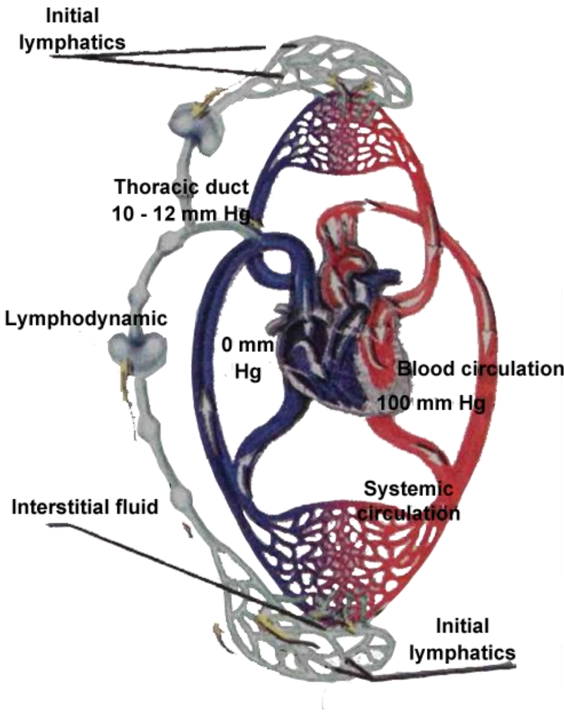
Motivation and goal



Functions of LS (among others):

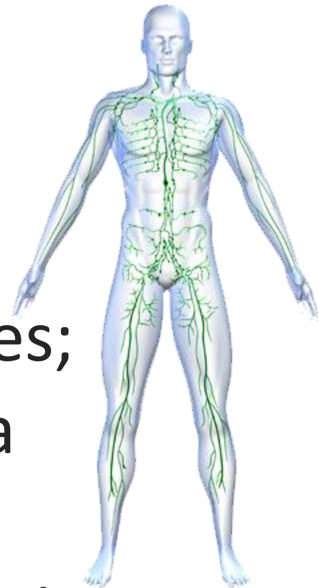
- transport: LS network is track for infection and drug distribution;
- immune: lymph nodes and lymphoid organs fight against infections;
- drainage: 10% of blood volume goes from cardio-vascular system (CVS) to LS

The goal of the work:

to create a quasi-onedimensional model of lymph flow through the whole lymphatic system with respect to features of LS

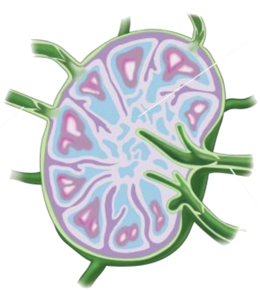
Anatomy and physiology

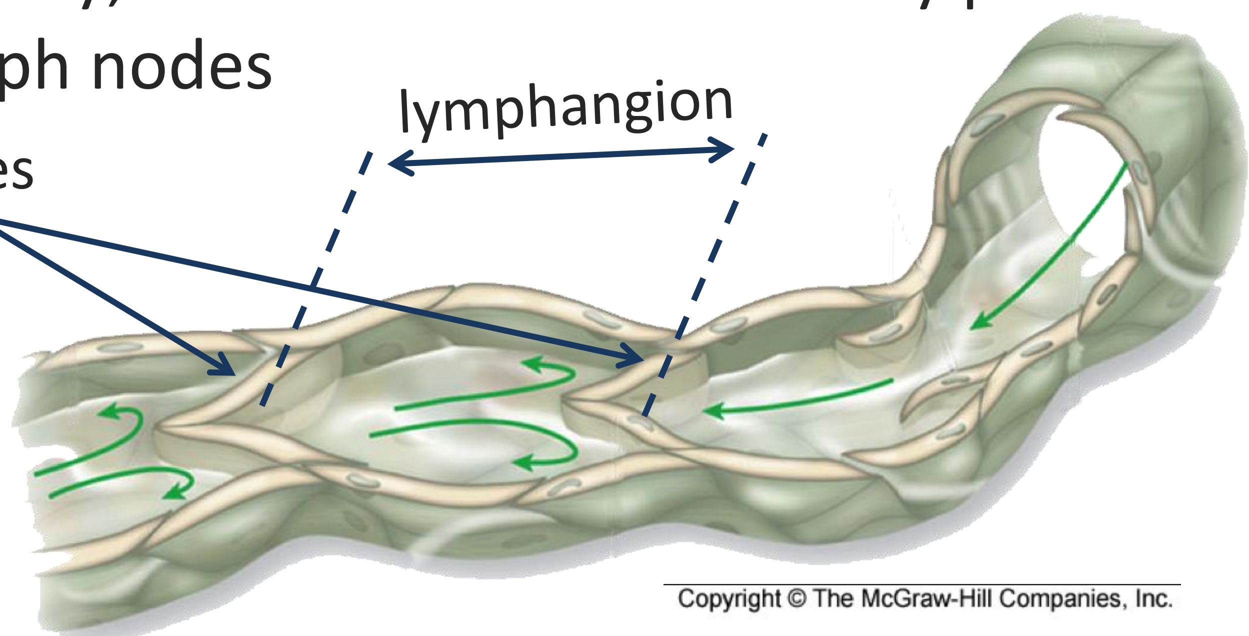
LS
- is not closed;
- starts with lymphatic capillaries;
- ends in upper vena cava;
- has no strict hierarchy;
- has numerous lymph nodes



Lymph flow
- low velocity;
- low pressure gradients;
- unidirectional

Lymph
- enters to the LS by portions





LS as a transport system

LS network:

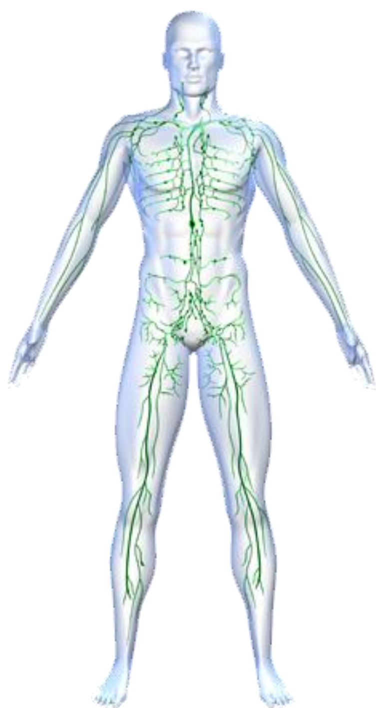
- lymphatic vessels;
- lymph nodes

Graph of the LS

Mechanisms of lymph propogation:

- global pressure gradient;
- valves;
- contractions

Mathematical model of lymph flow



Common values

- ∅ 1.5 – 2 mm (trunks, ducts);
- ∅ 3 – 5 μm to 1 – 2 mm (other);
- ∅ 20 – 200 μm (initial lymphatics);
- ∅ up to 5 mm (cisterna chyli)

- *l* = 35 – 40 cm (thoracic duct);
- *l* = 1.5 cm (right lymphatic duct).

Lymphangion length:

- 5 – 10 cm (thoracic duct);
- 12 – 15 mm (big vessels);
- 6 – 8 mm (smaller vessels);
- 2 – 3 mm (the first offsets);
- 1 – 2 mm (small inorganic).

Velocity:

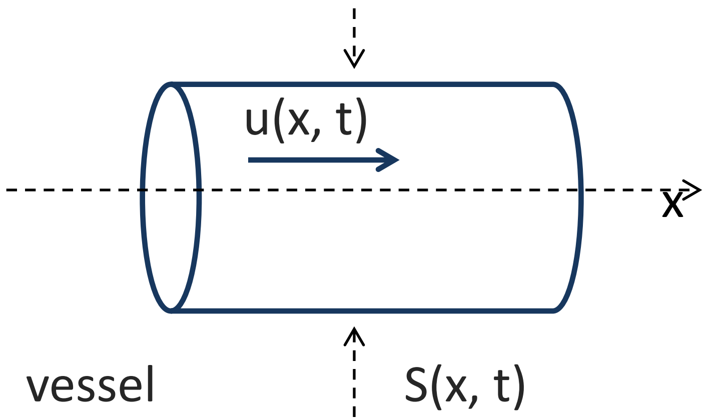
- 0.5 – 1 cm/s (trunks).

Contractions:

- velocity = 1 – 2 cm/s;
- period about 6 s;
- d(contracted)/d(relaxed) is about 40 % (diameter reduce is up to 60 %)

Mathematical models

Basic equations



$$\frac{\partial S}{\partial t}+\frac{\partial uS}{\partial x}=0,$$

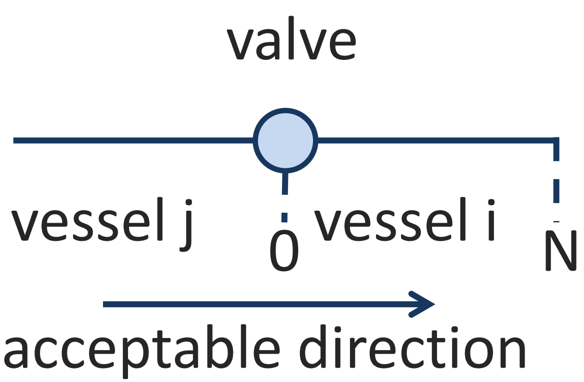
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=-8\pi\nu\frac{u}{S},$$

$$S=S(p)$$

S(x,t) – cross-section area, *u(x,t)* – lymph velocity, *p(x,t)* – pressure;

ρ – density, *ν* – viscosity; *x* – axial coordinate, *t* – time

Valves (nodes – for rare)

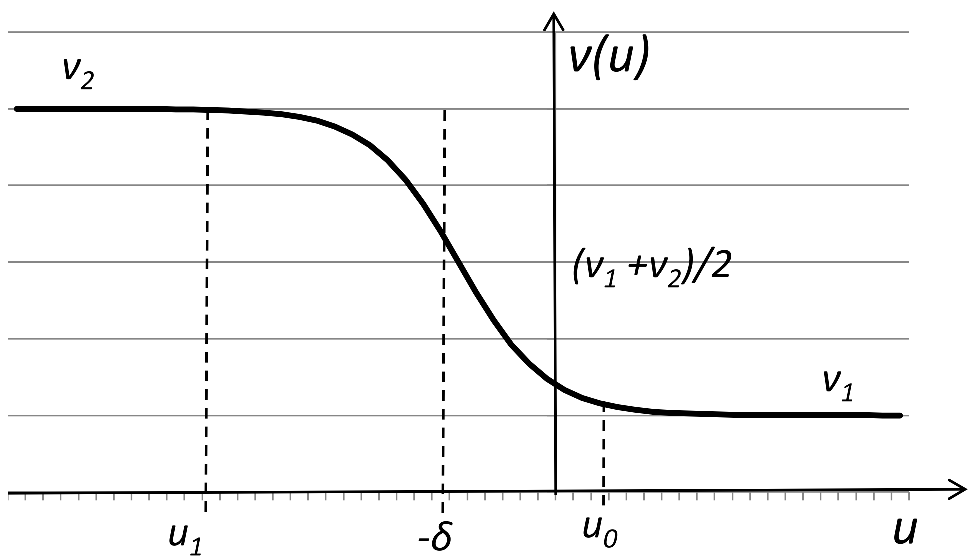


$$Q_i=\begin{cases}Q_j, & u_{i0}>0\\ 0, & u_{i0}<0\end{cases}$$

Valves (viscosity – for frequent)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = -8\pi\nu(u)\frac{u}{S}$$

$$\nu(u) = \frac{\exp\left(\alpha(u+\delta)\right)\nu_1 + \exp\left(-\beta(u+\delta)\right)\nu_2}{\exp\left(\alpha(u+\delta)\right) + \exp\left(-\beta(u+\delta)\right)}$$



Contractions

In state equation:

$$S(x,t) = S(p,x,t) = S_0 + \theta(p - p_0) - \theta A\cos\left(\frac{2\pi}{\lambda}(x - at)\right)$$

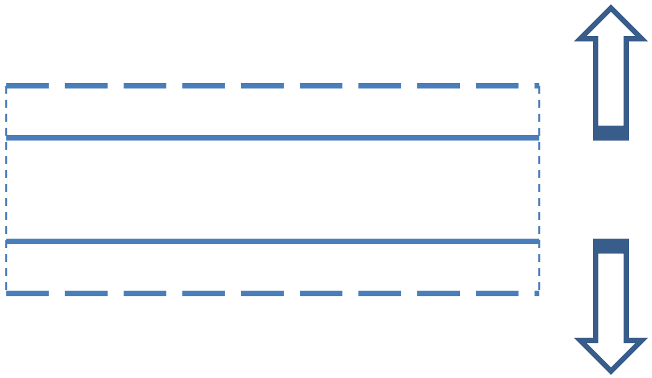
Contractions with *S=S(t)* (analytics)

$$\frac{\partial S}{\partial t} + \frac{\partial uS}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = -8\pi\nu\frac{u}{S},$$

$$S = S(t)$$

$$u(0,t) = u_0(t),\ p(0,t) = p_0(t),\ 0 < x < l$$



Solution:

$$u(x,t) = V(t)x + u_0(t),\ p(x,t) = P_A(t)\frac{x^2}{2} + P_B(t)x + p_0(t),\ 0 < x < l,\ t > 0$$

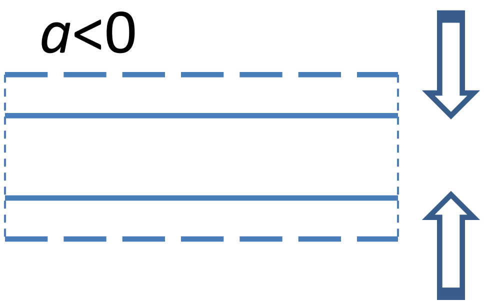
$$V(t) = -\frac{S'(t)}{S(t)},\ P_A(t) = -\rho\left[V'(t) + V^2(t) - k\frac{V(t)}{S(t)}\right],\ P_B(t) = -\rho\left[u_0'(t) + u_0(t)V(t) - k\frac{u_0(t)}{S(t)}\right],\ k = -8\pi\nu$$

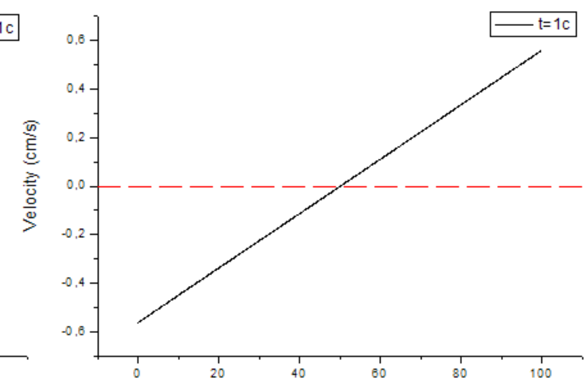
For $S = S_0 + at$ $\quad p(0,t) = p(l,t) = p_0(t),\, t > 0$

solution:
$$\begin{cases} u(x,t) = \dfrac{a}{S_0 + at}\left(\dfrac{l}{2} - x\right), \\ p(x,t) = -\dfrac{\rho a}{2(S_0 + at)^2}(2a + k)x^2 + \dfrac{\rho a}{2(S_0 + at)^2}(2a + k)lx + p_0(t) \end{cases}$$

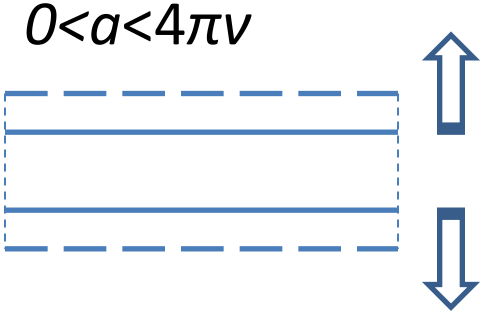
with critical pressure at central point $p_{cryt} = p_0(t) + \dfrac{\rho a(a - 4\pi\nu)l^2}{8S(t)^2}$

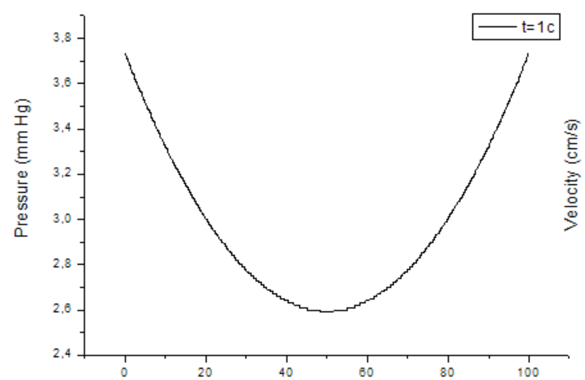
a<0

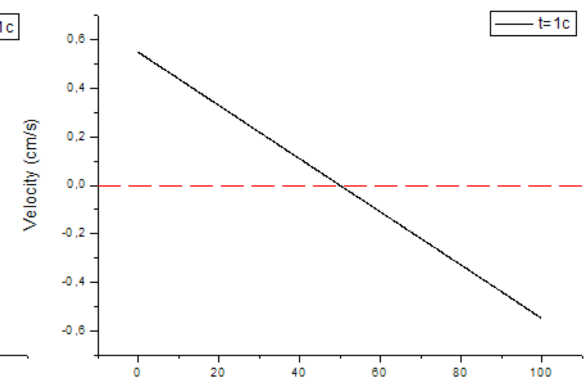




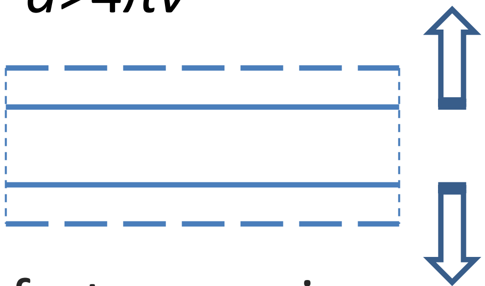
0<a<4πν



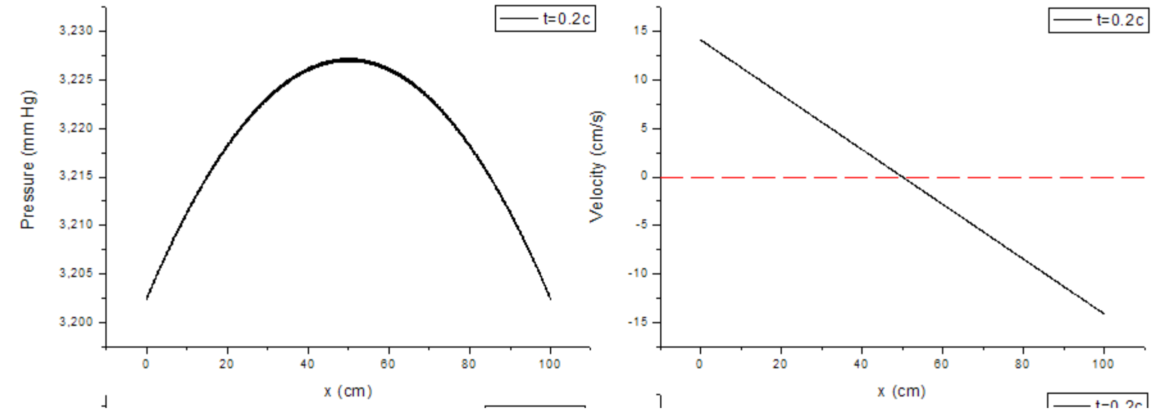




a>4πν



fast expansion



> *Makinde O.D.* Collapsible tube flow: a mathematical model // Rom. Journ. Phys, 2005, pp.493-506

S=S(t), ν=ν(u)

Solution: $u(x,t)=u_0(t)-\frac{S'}{S}x$

$$p(x,t)=p_0(t)+\rho\left[\frac{1}{2}\frac{S''}{S}-\left(\frac{S'}{S}\right)^2\right]x^2+\rho\left[u_0(t)\frac{S'}{S}-u_0'(t)\right]x+$$

$$8\pi\rho\frac{S'}{S^2}Ix-8\pi\rho\frac{u_0(t)}{S}I-8\pi\rho\frac{S'}{S^2}\int_{x_0}^{x}I\mathrm{d}x$$

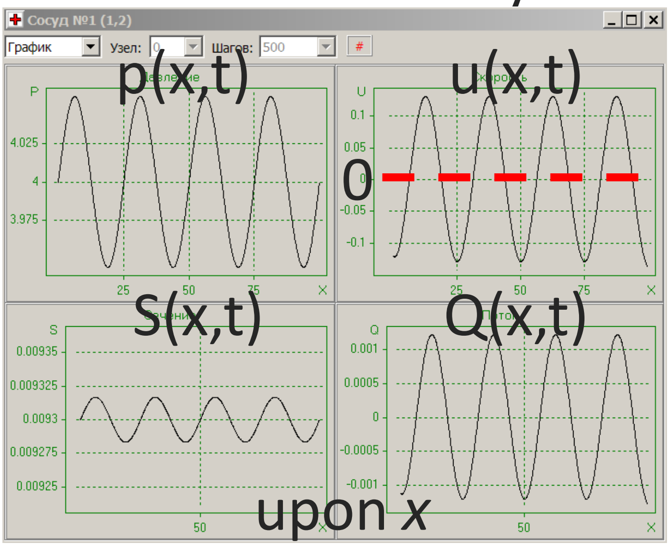
where $$I=I(x)=\int_{x_0}^{x}\nu(u)dx$$

Contractions with anisotropic viscosity (valves)

$$S = S(p,x,t) = S_0 + \theta(p - p_0) - \theta A \cos\left(\frac{2\pi}{\lambda}(x - at)\right)$$

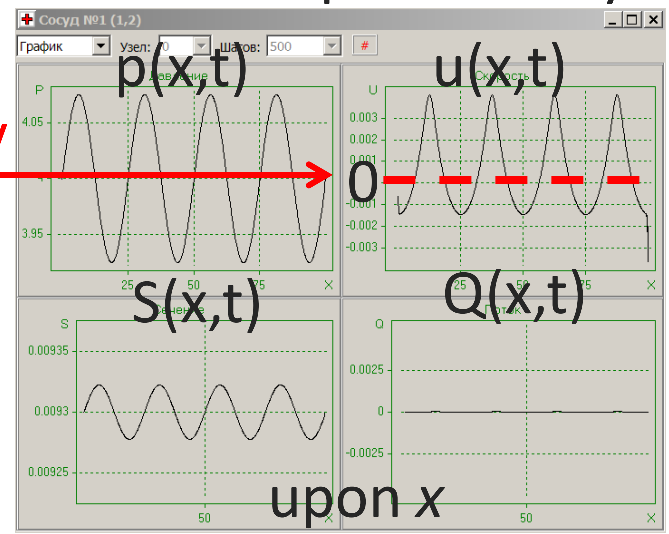
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = -8\pi\nu(u)\frac{u}{S} \qquad \nu(u) = \frac{\exp(\alpha(u+\delta))\nu_1 + \exp(-\beta(u+\delta))\nu_2}{\exp(\alpha(u+\delta)) + \exp(-\beta(u+\delta))}$$

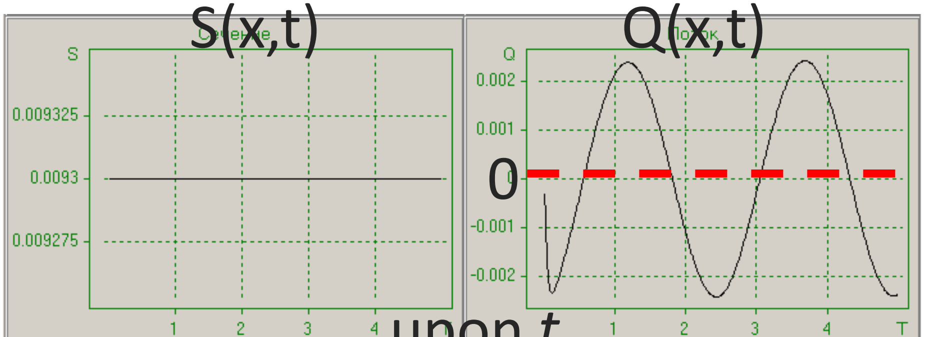
Normal viscosity

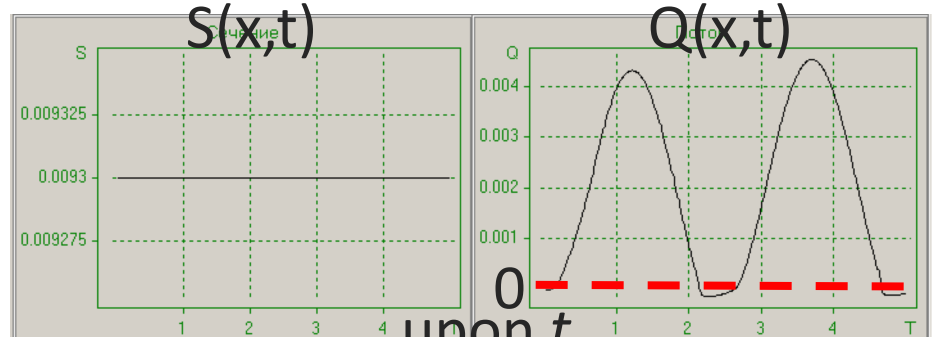


velocity change

Anisotropic viscosity



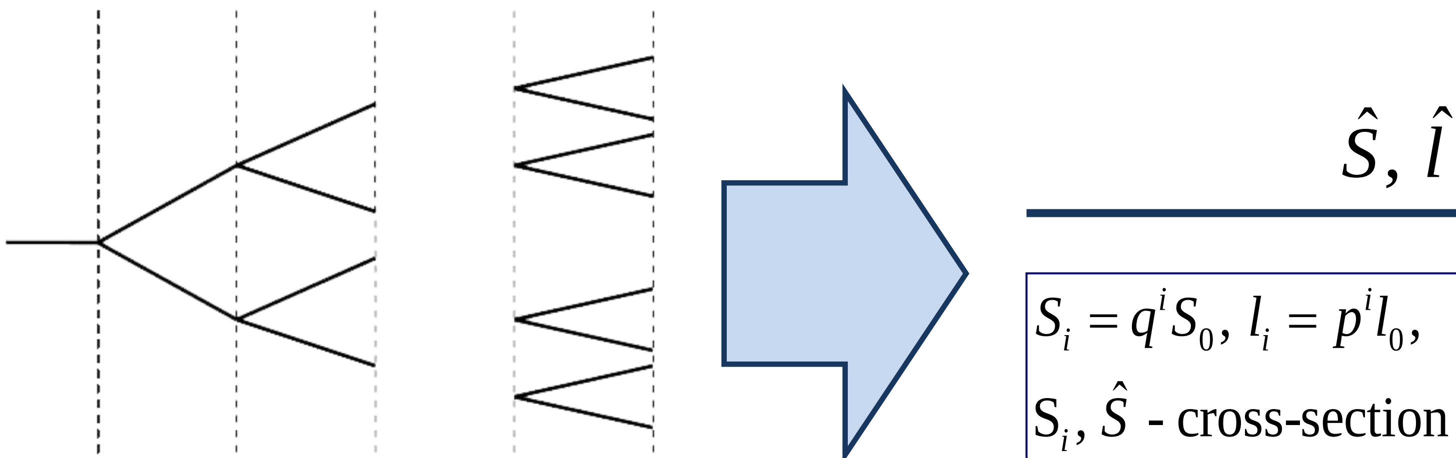




Flow become unidirectional (!)

Effective representation of initial lymphatics

Binary tree of initial lymphatics is substituted with one effective element with parameters that save flux, pressure gradient and lateral area of the net



$S_i = q^i S_0,\ l_i = p^i l_0,$

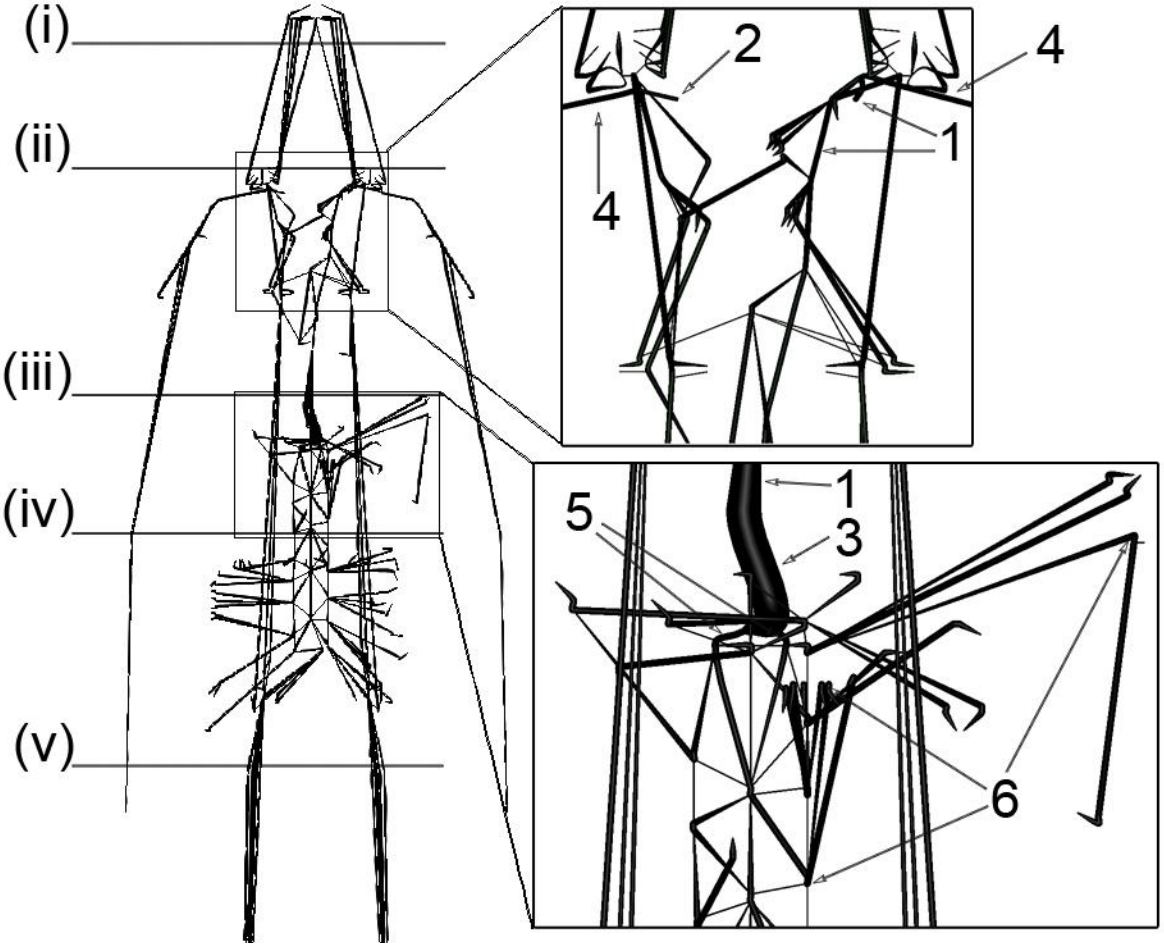
$S_i,\ \hat{S}$ - cross-section area,

$l_i,\ \hat{l}$ - length

$$\hat{S} = S_0 \left(\frac{\sum_{i=0}^{n} (2p\sqrt{q})^i}{\sum_{i=0}^{n} \left(\frac{p}{2\sqrt{q}} \right)^i} \right)^{2/5} \to S_0 \left(\frac{2q^2 - p}{2q^2 \left(1 - 2p\sqrt{q}\right)} \right)^{2/5}, \text{ when } 2\text{p}\sqrt{q} < 1,\ \frac{p}{2\sqrt{q}} < 1,\ n \to \infty$$

Graph of the LS

Graph of the LS



(i) head

(ii) neck

(iii) diaphragm

(iv) elbows

(v) groin

1 – thoracic duct

2 – right lymphatic duct

3 – cisterna chyli

4 – subclavian trunks

5 – lumbar trunks

6 – lymph nodes

- 161 lymph nodes in 46 regional groups
- 382 lymphatic vessels (7 trunks, 2 ducts, 157 effective)
- 25 valves (nodes)

Segmentation and parameters

Groups of arcs in the graph:

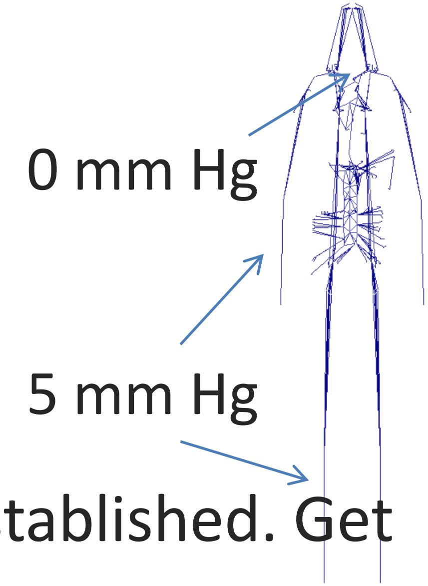
1. Trunks and ducts: active contractions.
2. Collectors: active contractions, valves (viscosity).
3. Effective vessels – basic equations.
4. Lymph nodes: active contractions.

Valves (nodes) are between arcs of the first group

Name	d (cm)	S (cm^2)
Effective vessels	0.02	0.00024
Collectors	0.107	0.009
Lumbar trunks	0.15	0.0177
Other trunks	0.1	0.0079
Ducts, lymph nodes	0.2	0.0314
Cisterna Chyli	0.4	0.1257

Calculations

Horizontal position



Goals: get presumable flux 0.023 ml/s ≈ 2 l/day

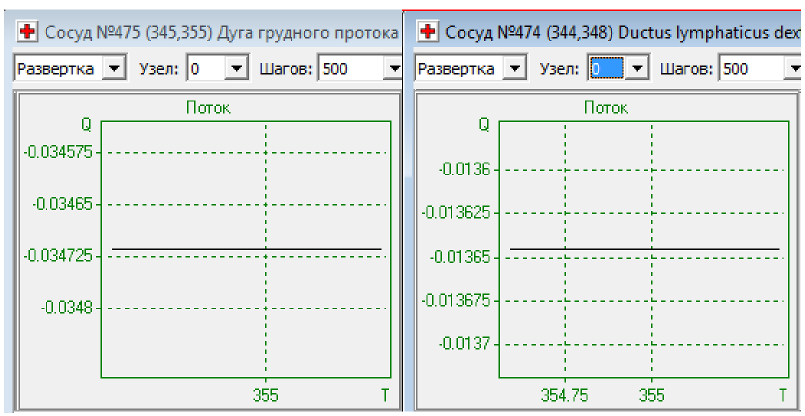
Initial data:

Pressure gradient = 5 mm Hg

No valves, no contractions, no gravity

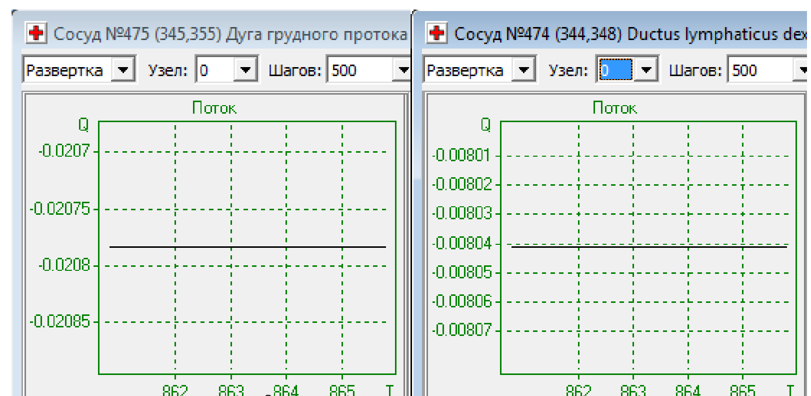
Calculations in CVSS program until steady flow is established. Get presumable flux by varying parameters of effective elements

Results:



Flux upon time

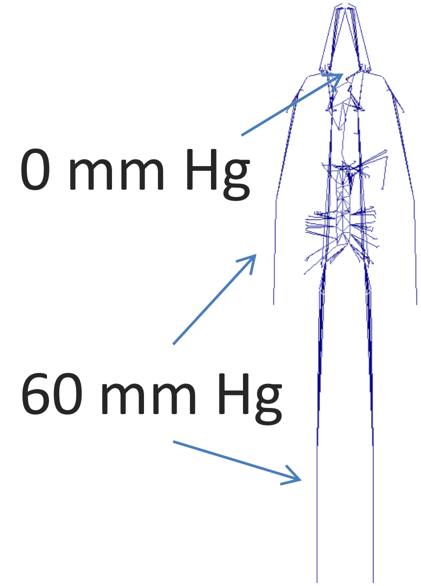
Output flux = 0.0483 ml/s



Flux upon time

Output flux = 0.0288 ml/s – is consistent with physiology value

Vertical position



Goals: obtain flux in the system

Initial data:

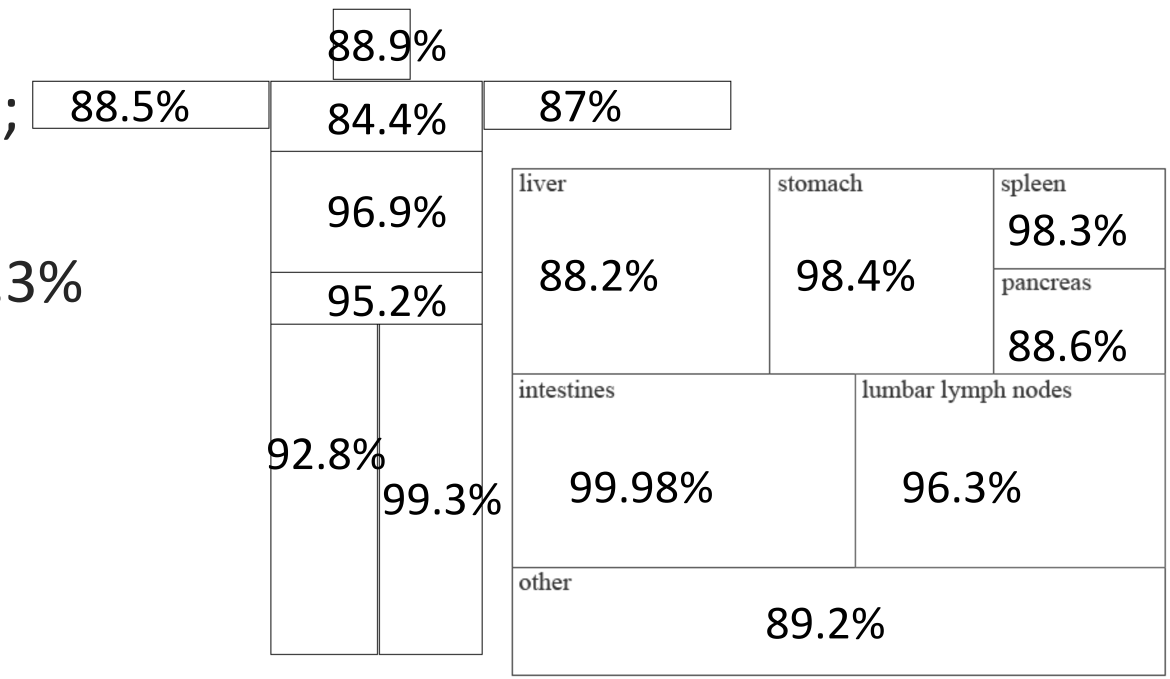
Pressure gradient (dp) = 60 mm Hg

Valves (nodes), no contractions, $|g| = 1000\ cm/s^2$

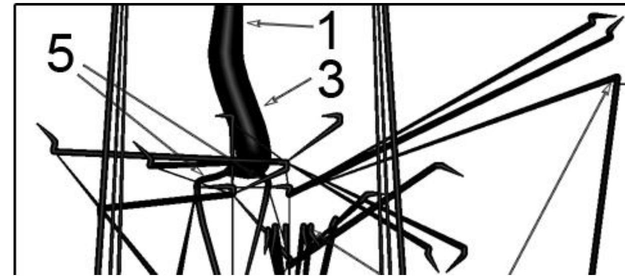
Results:

no flux in absence of any valves

- flux obtained;
- no flux with smaller dp;
- cycles;
- system fill on about 92.3%



Contractions of cisterna chyli



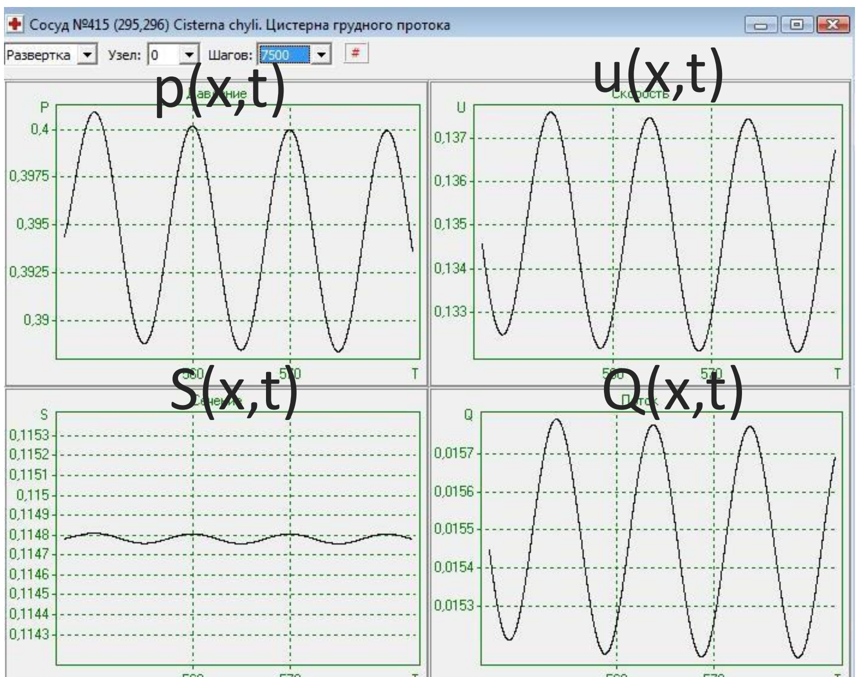
Goals: obtain flux in the system

Initial data:

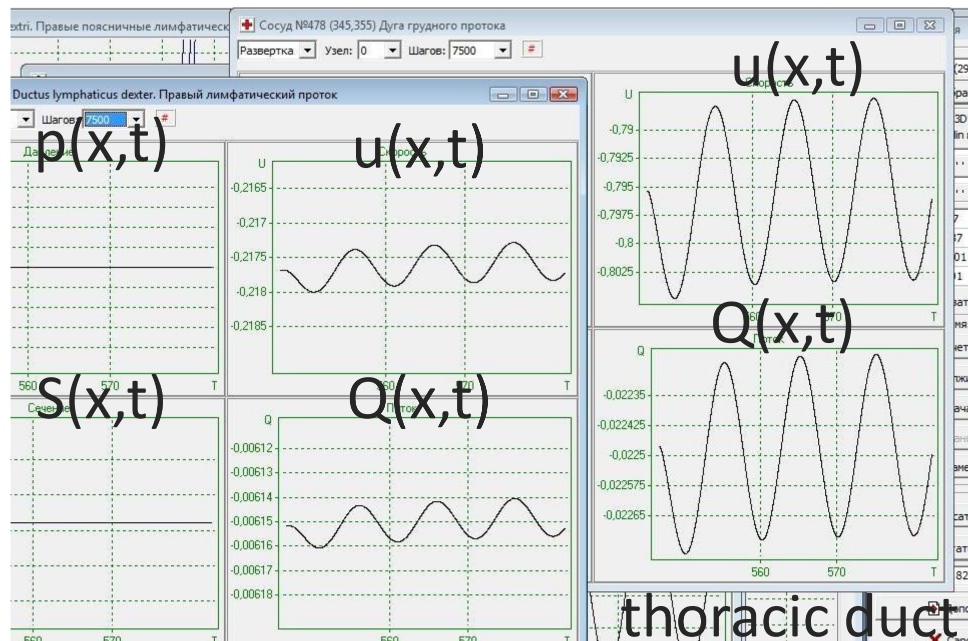
Pressure gradient (dp) = 5 mm Hg

Valves (nodes), valves (viscosity, v_2=1 cm^2/s), contractions of cisterna chyli (A=0.1 mm Hg, λ=10 cm, a=1 cm/s), no gravity

Results:



cisterna chyli



right lymphatic duct

Thank you for attention!